

DISTINGUISHING OFF-SHELL SUPERGRAVITIES WITH ON-SHELL PHYSICS

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ABSTRACT

We show that it is possible to distinguish between different off-shell completions of supergravity at the on-shell level. We focus on the comparison of the “new minimal” formulation of off-shell four-dimensional $N = 1$ supergravity with the “old minimal” formulation. We show that there are 3-manifolds which admit supersymmetric compactifications in the new-minimal formulation but which do not admit supersymmetric compactifications in other formulations. Moreover, on manifolds with boundary the new-minimal formulation admits “singleton modes” which are absent in other formulations.

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1 Introduction

The equations of motion of four-dimensional $N = 1$ supergravity can be obtained using the Euler-Lagrange equations applied to the Lagrangian ³

$$\mathcal{L}_{on-shell} = eR - 4i\bar{\psi}_\mu \Gamma^{\mu\nu\rho} D_\nu \psi_\rho \quad (1)$$

While this Lagrangian is invariant, up to a total derivative, under the transformation

$$\delta_\epsilon e_\mu{}^\nu = -2i\bar{\epsilon}\Gamma^\nu\psi_\mu, \quad \delta_\epsilon \psi_\mu = D_\mu \epsilon \quad (2)$$

these transformations do not close to the supersymmetry algebra unless the fields are taken to be on-shell. Indeed off-shell there are only six Bosonic degrees of freedom whereas there are twelve Fermionic degrees of freedom. Thus it is of interest to construct off-shell extensions of the supergravity Lagrangian.

The original motivations for studying off-shell completions of supergravity were to ensure that supersymmetry remains a valid symmetry at the quantum level as well as to facilitate the proof of non-renormalization theorems. A natural starting point for supergravity is the geometrical analysis of four-dimensional $N = 1$ superspace. Four-dimensional $N = 1$ superfields carry reducible supersymmetry multiplets. Therefore additional constraints need to be imposed to truncate the superfields. These are then combined with the torsion and Bianchi identities to solve for the independent fields. It turns out that there are various ways to do this and hence there are several different off-shell formulations. The predominant view of these various off-shell completions is that they are all equivalent on-shell. The purpose of this paper is to show that while this is locally true, it is globally false.

In this paper we will explore some novel aspects of an off-shell formulation of four-dimensional $N = 1$ supergravity known as “new minimal supergravity” (NMS) [1]. We show that NMS admits Killing spinors on manifolds which are not supersymmetric in other formulations such as the “old-minimal” formulation [2, 3, 4, 5, 6]. Moreover, when formulated on a manifold with boundary certain gauge modes of the auxiliary fields of NMS can become dynamical (depending on boundary conditions).

From a string theory perspective off-shell formulations are often viewed as unnecessary luxuries since one is simply viewing supergravity as a low energy effective theory which reproduces the correct on-shell physics. Our results call that point of view into question. A key motivation for this study was the desire to formulate 11-dimensional supergravity on Spin^c manifolds

³Here $\mu, \nu, \dots = 0, 1, 2, 3$ are world indices and an underline denotes the tangent frame. We use the $(-, +, +, +)$ signature with $\{\Gamma_\mu, \Gamma_\nu\} = 2g_{\mu\nu}$, $\Gamma_5 = \frac{1}{24}\epsilon^{\mu\nu\lambda\rho}\Gamma_{\mu\nu\lambda\rho}$ and $\epsilon^{0123} = 1$.

(see the discussion section below). Indeed NMS contains an auxiliary gauge field which allows one to define the theory (off-shell) on Spin^c manifolds. Unfortunately very little is known about off-shell completions of 11-dimensional supergravity. Indeed it is generally believed that an infinite number of auxiliary fields are required (although in some circumstances one could consider a finite collection of auxiliary fields which do not entirely close the algebra [7]). Our results raise the important question of whether different off-shell formulations of M-theory could be physically inequivalent.

2 Old and New Minimal Supergravity

The most familiar off-shell completion of $N = 1$ four-dimensional supergravity is the so-called old-minimal formulation which includes two real scalar fields M and N along with a one-form b [2, 3, 4, 5, 6]. The action is simply

$$\mathcal{L}_{old-minimal} = \mathcal{L}_{on-shell} - \frac{1}{2}eM^2 - \frac{1}{2}eN^2 + \frac{1}{2}b \wedge \star b \quad (3)$$

and there are twelve Bosonic and twelve Fermionic degrees of freedom off-shell. Clearly these new fields do not alter the theory in any non-trivial way. However one does find that the supersymmetry algebra closes off-shell (along with appropriate modifications to the supertransformation rules to include the auxiliary fields).

In [1] Sohnius and West gave an off-shell formulation of four-dimensional $N = 1$ supergravity which, in addition to the graviton and gravitini, includes an auxiliary 1-form $A = A_\mu dx^\mu$ and a 2-form $B = \frac{1}{2}B_{\mu\nu}dx^\mu \wedge dx^\nu$. The Lagrangian is

$$\mathcal{L}_{NMS} = eR - 4ie\bar{\psi}_\mu \Gamma^{\mu\nu\rho} \mathcal{D}_\nu^+ \psi_\rho - 6V \wedge \star V - 4A \wedge dB \quad (4)$$

where

$$\begin{aligned} V &= -\frac{1}{2} \star (dB - i\bar{\psi}_\nu \Gamma_\lambda \psi_\rho dx^\nu \wedge dx^\lambda \wedge dx^\rho) \\ \mathcal{D}_\mu^+ \psi_\nu &= D_\mu \psi_\nu + A_\mu \Gamma_5 \psi_\nu \\ \omega_{\mu\lambda\rho} &= \omega_{\mu\lambda\rho}^{\text{Levi-Civita}} - i(\bar{\psi}_\lambda \Gamma_\mu \psi_\rho + \bar{\psi}_\mu \Gamma_\lambda \psi_\rho - \bar{\psi}_\mu \Gamma_\rho \psi_\lambda) \end{aligned} \quad (5)$$

The equation of motion for A sets

$$V = 0 \quad (6)$$

and then the equation of motion for B determines that A is a flat connection

$$dA = 0 \quad (7)$$

The particular choice $A = 0$ then leads to the usual equations of motion for the graviton and gravitini.

In addition to diffeomorphisms the theory is invariant under the local supersymmetry transformation

$$\begin{aligned}
\delta_\epsilon e_\mu{}^\nu &= -2i\bar{\epsilon}\Gamma^\nu\psi_\mu \\
\delta_\epsilon\psi_\mu &= \mathcal{D}_\mu^+\epsilon - V_\mu\Gamma_5\epsilon + \frac{1}{2}\Gamma_\mu{}^\nu\Gamma_5V_\nu\epsilon \\
\delta_\epsilon B_{\mu\nu} &= 4i\bar{\epsilon}\Gamma_{[\mu}\psi_{\nu]} \\
\delta_\epsilon A_\mu &= -2i\bar{\epsilon}\Gamma_5\Gamma_\mu\Gamma^{\nu\lambda}\left(\mathcal{D}_\nu^+\psi_\lambda + 3\Gamma_5\psi_\nu V_\lambda + \frac{3}{2}\Gamma_\nu{}^\rho\psi_\lambda V_\rho\right)
\end{aligned} \tag{8}$$

However the crucial difference [8] between old and new minimal supergravity is that NMS is also invariant under a local chiral rotation

$$\begin{aligned}
\delta_\chi e_\mu{}^\nu &= 0 \\
\delta_\chi\psi_\mu &= -\chi\Gamma_5\psi_\mu \\
\delta_\chi B &= 0 \\
\delta_\chi A &= d\chi.
\end{aligned} \tag{9}$$

which is broken in old minimal by the supersymmetry transformation rules. The Lagrangian also has a trivial gauge transformation which only acts on B

$$\delta_\lambda B = d\lambda. \tag{10}$$

It is easy to check that all these symmetries commute with each other

$$[\delta_\epsilon, \delta_\chi] = [\delta_\epsilon, \delta_\lambda] = [\delta_\chi, \delta_\lambda] = 0 \tag{11}$$

provided that the supersymmetry generator is also taken to transform under local chiral rotations.

Under supersymmetry the action is not invariant but rather transforms into a boundary term. To quadratic order in the Fermions one finds

$$\begin{aligned}
\delta_\epsilon S &= \int_{\mathcal{M}} \delta_\epsilon \mathcal{L}_{NMS} \\
&= -2i \int_{\partial\mathcal{M}} \sqrt{-h}(h^{\rho\nu}n^\mu + h^{\rho\mu}n^\nu - 2h^{\mu\nu}n^\rho)D_\rho(\bar{\epsilon}\Gamma_\mu\psi_\nu) \\
&\quad + 4i \int_{\partial\mathcal{M}} \bar{\epsilon} \left(\Gamma_5\Gamma_\lambda\mathcal{D}_\nu^-\psi_\rho + \Gamma_\lambda V_\nu\psi_\rho + \frac{1}{2}\Gamma_\nu{}^\tau\Gamma_\lambda V_\tau\psi_\rho \right) dx^\lambda \wedge dx^\nu \wedge dx^\rho
\end{aligned} \tag{12}$$

where n^μ is the unit inward pointing normal vector to the boundary, $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ is the induced metric and

$$\mathcal{D}_\nu^- \psi_\rho = D_\nu \psi_\rho - \Gamma_5 A_\nu \psi_\rho \quad (13)$$

is the anti-chiral covariant derivative, *i.e.* it corresponds to gauging chiral rotations with the opposite choice of Γ_5 . The fact that $\delta_\epsilon \mathcal{L}$ is no longer chirally invariant seems odd but can be verified by noting that, due to the final Chern-Simons term, the Lagrangian is not exactly chirally invariant either

$$\delta_\chi \mathcal{L}_{NMS} = -4d\chi \wedge dB = -4d(\chi dB) \quad (14)$$

The failure of chiral symmetry in $\delta_\epsilon \mathcal{L}$ is then necessary to account for the variation of $\delta_\chi \mathcal{L}$ under supersymmetry since it must be true that

$$\delta_\chi \delta_\epsilon \mathcal{L}_{NMS} = \delta_\epsilon \delta_\chi \mathcal{L}_{NMS} \quad (15)$$

which one can readily check is indeed the case with these boundary terms.

However we can correct for this by adding the total derivative term

$$\mathcal{L}_{bdry} = 4d(A \wedge B) \quad (16)$$

to the Lagrangian. This has the effect of changing the Lagrangian to

$$\mathcal{L}'_{NMS} = eR - 4ie\bar{\psi}_\mu \Gamma^{\mu\nu\rho} \mathcal{D}_\nu^+ \psi_\rho - 6V \wedge \star V - 4dA \wedge B \quad (17)$$

Clearly \mathcal{L}'_{NMS} is invariant under chiral rotations, with no boundary terms. This implies that the boundary terms must be invariant under chiral rotations. The variation of \mathcal{L}_{bdry} under supersymmetry is

$$\delta_\epsilon \mathcal{L}_{bdry} = 4d(\delta_\epsilon A \wedge B - 2i\bar{\epsilon} A_\lambda \Gamma_\nu \psi_\rho dx^\lambda \wedge dx^\nu \wedge dx^\rho) \quad (18)$$

The second term converts the \mathcal{D}_μ^- into a \mathcal{D}_μ^+ derivative. Thus the variation of the improved action under supersymmetry is

$$\begin{aligned} \delta_\epsilon S' &= \int_{\mathcal{M}} \delta_\epsilon \mathcal{L}_{NMS} + \int_{\partial\mathcal{M}} \delta_\epsilon \mathcal{L}_{bdry} \\ &= -2i \int_{\partial\mathcal{M}} \sqrt{-h} (h^{\rho\nu} n^\mu + h^{\rho\mu} n^\nu - 2h^{\mu\nu} n^\rho) D_\rho (\bar{\epsilon} \Gamma_\mu \psi_\nu) \\ &\quad + 4i \int_{\partial\mathcal{M}} \bar{\epsilon} \left(\Gamma_5 \Gamma_\lambda \mathcal{D}_\nu^+ \psi_\rho + \Gamma_\lambda V_\nu \psi_\rho + \frac{1}{2} \Gamma_\nu{}^\tau \Gamma_\lambda V_\tau \psi_\rho \right) dx^\nu \wedge dx^\rho \wedge dx^\lambda \\ &\quad + 4 \int_{\partial\mathcal{M}} \delta_\epsilon A \wedge B \end{aligned} \quad (19)$$

which is indeed invariant under chiral rotations.

2.1 Supersymmetry in the presence of boundaries

It is well known that if supergravity is placed on a manifold with a boundary then at least half of the supersymmetries will be broken. If $\partial\mathcal{M} \neq 0$ then the action obtained from \mathcal{L}' is not invariant under supersymmetry. However by adding suitable boundary terms this can be corrected. These boundary terms are used to set-up a well-posed boundary value problem and also to preserve half of the supersymmetries. In the case of eleven-dimensional supergravity this has been done in [9] and we wish to follow a similar analysis for NMS, although we will restrict our attention to quadratic terms in the Fermions.

The first step to including boundaries is to add the Gibbons-Hawking term to make the pure gravitational variational problem well posed

$$S_{GH} = 2 \int_{\partial\mathcal{M}} \sqrt{-h} K \quad (20)$$

where $K_{\mu\nu} = h_{\mu\lambda} h_{\nu\rho} D^\lambda n^\rho$ is the extrinsic curvature. With this term in place one finds that the variation of the standard Einstein-Hilbert plus Gibbons-Hawking term results in the boundary term

$$\delta_g(S_{EH} + S_{GH}) = - \int_{\partial\mathcal{M}} \sqrt{-h} (K_{\mu\nu} - g_{\mu\nu} K) \delta g^{\mu\nu} \quad (21)$$

which is required to cancel with any additional stress-energy tensor that is localized to the boundary (which in our case vanishes).

Following [9] we need to add a Fermionic boundary term. Let us first recall the case of the familiar on-shell supergravity. Varying the Fermionic term gives the equations of motion plus the boundary term

$$\delta_\psi S = 4i \int_{\partial\mathcal{M}} \sqrt{-h} \bar{\psi}_{\mu'} \Gamma^{\mu'\nu'} \Gamma^n \delta\psi_{\nu'} \quad (22)$$

where $\Gamma^n = n^\mu \Gamma_\mu$ and μ', ν' are the coordinates tangential to the boundary. To cancel this one adds the term [10]

$$S_{LM} = 2i\xi \int_{\partial\mathcal{M}} \sqrt{-h} \bar{\psi}_{\mu'} \Gamma^{\mu'\nu'} \psi_{\nu'} \quad (23)$$

where $\xi = \pm 1$. Variation of this term gives

$$\delta_\psi S_{LM} = 4i\xi \int_{\partial\mathcal{M}} \sqrt{-h} \bar{\psi}_{\mu'} \Gamma^{\mu'\nu'} \delta\psi_{\nu'} \quad (24)$$

Thus a suitable boundary condition is $\psi_{\mu'} = \xi \Gamma^n \psi_{\mu'}$.

In NMS we again encounter the boundary term (22) however we cannot simply add (23) since this term is not invariant under chiral rotations. More properly it doesn't even make sense as ψ_μ is a section of a chiral spinor bundle whereas $C\Gamma^{\mu'\nu'}$ is a map between the chiral and anti-chiral spinor bundles.

If we choose a boundary condition where $A = d\Phi_A$ on $\partial\mathcal{M}$ then we can add the boundary term

$$S_{\partial\psi} = 2i\xi \int_{\partial\mathcal{M}} \sqrt{-h} \bar{\psi}_{\mu'} \Gamma^{\mu'\nu'} e^{2\Phi_A \Gamma_5} \psi_{\nu'} \quad (25)$$

i.e. we can map NMS to on-shell supergravity by a chiral gauge rotation. (Note that this boundary condition implies that the gauge field A is trivial on the boundary and this is not the case in general.) We now find the Fermionic boundary condition

$$\Gamma^n e^{\Phi_A \Gamma_5} \psi_{\nu'} = \xi e^{\Phi_A \Gamma_5} \psi_{\nu'} \quad (26)$$

Our next task is to show that

$$S = S_{GH} + S_{\partial\psi} + \int_{\mathcal{M}} \mathcal{L}' \quad (27)$$

is indeed supersymmetric with a well posed variational problem when we impose the boundary conditions

$$\Gamma^n e^{2\Phi_A \Gamma_5} \psi_{\nu'} = \xi \psi_{\nu'} , \quad A = d\Phi_A, \quad V = 0 \quad (28)$$

on $\partial\mathcal{M}$. We derived the boundary terms by ensuring a well-posed boundary value problem for the metric and ψ_μ . Thus it remains to check that variations of the form $\delta A = d\delta\Phi_A$ on the boundary are well posed, *i.e.* that they do not over constrain the system. There are two sources for these variations, boundary terms from the bulk $d\delta A \wedge B$ term and also terms that arise directly from varying $S_{\partial\psi}$. Putting these together we find

$$\begin{aligned} \delta_{\Phi_A} S &= 4 \int_{\partial\mathcal{M}} d\delta\Phi_A \wedge B + 4i\xi \int_{\partial\mathcal{M}} \sqrt{-h} \delta\Phi_A \bar{\psi}_{\mu'} \Gamma^{\mu'\nu'} \Gamma_5 e^{2\Phi_A \Gamma_5} \psi_{\nu'} \\ &= -4 \int_{\partial\mathcal{M}} \delta\Phi_A \wedge dB - i\sqrt{-h} \delta\Phi_A \bar{\psi}_{\mu'} \Gamma^{\mu'\nu'} \Gamma_5 \Gamma^n \psi_{\nu'} \\ &= -4 \int_{\partial\mathcal{M}} \delta\Phi_A \wedge *V \\ &= 0 \end{aligned} \quad (29)$$

Next we examine the variation of the action under supersymmetry. We already know that the bulk is supersymmetric. We would like to show that the variation of the additional boundary terms $S_{GH} + S_{\partial\psi}$ cancels (19). Of course the boundary condition on the Fermions breaks half of the supersymmetries, leaving only those with $\Gamma^n e^{\Phi_A \Gamma_5} \epsilon = \xi e^{\Phi_A \Gamma_5} \epsilon$. Therefore we compute

$$\delta_\epsilon(S_{EH} + S_{GH}) = 4i \int_{\partial\mathcal{M}} \sqrt{-h} \bar{\epsilon} (K^{\mu'\nu'} - h^{\mu'\nu'} K) \Gamma_{\mu'} \psi_{\nu'} \quad (30)$$

and (using the boundary conditions)

$$\begin{aligned} \delta_\epsilon S_{\partial\psi} &= 4i\xi \int_{\partial\mathcal{M}} \sqrt{-h} \left(\mathcal{D}_{\mu'}^+ \bar{\epsilon} \Gamma^{\mu'\nu'} e^{2\Phi_A \Gamma_5} \psi_{\nu'} + \delta_\epsilon \Phi_A \bar{\psi}_{\mu'} \Gamma^{\mu'\nu'} e^{2\Phi_A \Gamma_5} \Gamma_5 \psi_{\nu'} \right) \\ &= 4i \int_{\partial\mathcal{M}} \sqrt{-h} \left(\mathcal{D}_{\mu'}^+ \bar{\epsilon} \Gamma^{\mu'\nu'} \Gamma^n \psi_{\nu'} + \delta_\epsilon \Phi_A \bar{\psi}_{\mu'} \Gamma^{\mu'\nu'} \Gamma_5 \Gamma^n \psi_{\nu'} \right) \\ &= -4i \int_{\partial\mathcal{M}} \sqrt{-h} \left(\bar{\epsilon} \Gamma^{\mu'\nu'} \Gamma^n \mathcal{D}_{\mu'}^+ \psi_{\nu'} + \bar{\epsilon} \Gamma^{\mu'\nu'} D_{\mu'} n_\lambda \Gamma^\lambda \psi_{\nu'} \right) \\ &\quad + 4 \int_{\partial\mathcal{M}} \delta_\epsilon \Phi_A \wedge dB \\ &= -4i \int_{\partial\mathcal{M}} \sqrt{-h} \left(\bar{\epsilon} \Gamma^{\mu'\nu'} \Gamma^n D_{\mu'}^+ \psi_{\nu'} + \bar{\epsilon} (K^{\mu'\nu'} - K h^{\mu'\nu'}) \Gamma_{\mu'} \psi_{\nu'} \right) \\ &\quad - 4 \int_{\partial\mathcal{M}} \delta_\epsilon A \wedge B \end{aligned} \quad (31)$$

Here we have used the identity

$$\Gamma^{\mu'\nu'} D_{\mu'} n_\lambda \Gamma^\lambda = K_{\mu'\lambda'} \Gamma^{\mu'\nu'} \Gamma^{\lambda'} = (K^{\mu'\nu'} - K h^{\mu'\nu'}) \Gamma_{\mu'} \quad (32)$$

One can now see that these terms precisely cancel the terms in (19).

Note that we never had to deduce what $\delta_\epsilon \Phi_A$ was from $\delta_\epsilon A$. However we have assumed that $A = d\Phi_A$ and $V = 0$ on the boundary and these conditions impose constraints on the Fermions on the boundary (which are certainly satisfied if the Fermions are on-shell).

Returning to the case of a general A we see that if it is non-trivial on $\partial\mathcal{M}$ then there is no boundary term that we can write down that will cancel the Fermion boundary variation. Thus in these cases supersymmetry is broken by the presence of a boundary.

3 Higher-Dimensional Supergravity and an On-Shell Variant of New-Minimal

Ultimately we are interested in extending our analysis to the ten and eleven-dimensional supergravities associated with string theory and M-theory. As mentioned above, no off-shell completions are known for these theories and it is generally believed that if any such formulations exist then they must have an infinite number of auxiliary fields.

However one can see that the type of physics explored here can be extended in part to other supergravities. Suppose that there is a supergravity Lagrangian \mathcal{L}_{sugra} which is also invariant under a global symmetry (up to a boundary term). Then we can make this symmetry local in the usual manner by replacing covariant derivatives with gauge-covariant derivatives

$$D_\mu \rightarrow \mathcal{D}_\mu = D_\mu + \mathcal{A}_\mu \quad (33)$$

where \mathcal{A}_μ is the appropriate gauge connection. In this way we obtain the new Lagrangian

$$\mathcal{L}_\mathcal{A} = \mathcal{L}_{sugra}(D_\mu \rightarrow \mathcal{D}_\mu) \quad (34)$$

Next we must arrange for $\mathcal{L}_\mathcal{A}$ to be supersymmetric. One sees that if we choose

$$\delta_\epsilon \mathcal{A}_\mu = 0 \quad (35)$$

then the variation of the action, ignoring boundary terms, must be of the form

$$\delta_\epsilon \mathcal{L}_\mathcal{A} = \text{Tr}(\mathcal{F} \wedge \Omega) \quad (36)$$

where \mathcal{F} is the gauge-invariant field strength of \mathcal{A} . To cancel such a term we need only invent a new form field \mathcal{B} with

$$\delta_\epsilon \mathcal{B} = \Omega \quad (37)$$

so that

$$\mathcal{L}'_\mathcal{A} = \mathcal{L}_\mathcal{A} - \text{Tr}(\mathcal{F} \wedge \mathcal{B}) \quad (38)$$

is supersymmetric, up to possible boundary terms.

In this way we have arrived a form of supergravity that is similar to NMS. Of course the supersymmetry algebra is not closed off-shell, indeed we have merely added a supersymmetry singlet \mathcal{A} , along with a non-singlet field \mathcal{B} . Thus these modifications may lead to problems in the quantum theory.

For example we could consider $N = 1$ supergravity in four dimensions and gauge the chiral $U(1)$ symmetry that new-minimal exploits. Following the above procedure we arrive at the Lagrangian

$$\tilde{\mathcal{L}} = eR - 4ie\bar{\psi}_\mu \Gamma^{\mu\nu\rho} \mathcal{D}_\nu^+ \psi_\rho - 4F \wedge B \quad (39)$$

which is invariant under

$$\begin{aligned}
\delta_\epsilon e_\mu{}^\nu &= -2i\bar{\epsilon}\Gamma^\nu\psi_\mu \\
\delta_\epsilon\psi_\mu &= \mathcal{D}_\mu^+\epsilon \\
\delta_\epsilon B_{\mu\nu} &= 4i\bar{\epsilon}\Gamma_{[\mu}\psi_{\nu]} \\
\delta_\epsilon A_\mu &= 0
\end{aligned} \tag{40}$$

up to boundary terms. This is similar to, but not identical to, new-minimal with $V = 0$. In addition it is easy to see that this theory can be made supersymmetric on a manifold with boundary using the same boundary terms and conditions that we used for new-minimal.

4 Supersymmetric Compactifications

We can classify all supersymmetric compactifications of NMS supergravity (and also its variant (39)) of the form $\mathbf{R}^{4-d} \times \mathcal{M}_d$ with $d = 1, 2, 3$ and \mathcal{M}_d compact without boundary. Our first condition on the manifold \mathcal{M}_d , apart from compactness, is that it admits some kind of spinor and hence must be orientable (we will not consider the possibility of pinors here). The existence of a Killing spinor ϵ such that $\mathcal{D}_\mu^+\epsilon = 0$ also implies that

$$0 = [\mathcal{D}_\mu^+, \mathcal{D}_\nu^+]\epsilon = \frac{1}{4}R_{\mu\nu\lambda\rho}\Gamma^{\lambda\rho}\epsilon + F_{\mu\nu}\Gamma_5\epsilon \tag{41}$$

Since $F = dA = 0$ on-shell this implies that \mathcal{M}_d is Ricci-flat and hence also Riemann flat since $d = 1, 2, 3$.

For $d = 1, 2$ the only possible internal manifolds are tori. These can clearly be made supersymmetric. On the other hand we could also turn on a non-trivial $A = \frac{\alpha}{R_3}dx^3$ where x^3 is taken to be periodic with period $2\pi R_3$. The Killing spinors take the form

$$\epsilon = e^{-\alpha\frac{x^3}{R_3}\Gamma_5}\epsilon_0 \tag{42}$$

with ϵ_0 a constant spinor. Only if $\alpha \in \mathbf{Z}$ do we find a single valued spinor. In this case the supercurrent will not be single valued so that the variation of the Lagrangian is not exact and hence the action not invariant. Thus a generic A will apparently break all the supersymmetries. However we can undo the damage if we simply change the boundary conditions of the gravitino to $\psi_\mu(x^3 + 2\pi R_3) = e^{-2\pi\alpha\Gamma_5}\psi_\mu(x^3)$. In this case the supercurrent will be single valued and hence the variation of the Lagrangian is an exact form and the action invariant.

4.1 Compactification to one-dimension

The case of $d = 3$ is more interesting. There are in fact six compact orientable Riemann flat three-manifolds called Bieberbach manifolds (for example see [11]). They are all obtained as quotients of \mathbf{R}^3 by some freely acting group G and can be identified by their holonomies

$$\mathcal{H}(\mathcal{M}_3) = \mathbf{1} , \mathbf{Z}_2 , \mathbf{Z}_3 , \mathbf{Z}_4 , \mathbf{Z}_6 , \mathbf{Z}_2 \times \mathbf{Z}_2 \quad (43)$$

The first case is of course that of the torus $\mathbf{T}^3 = \mathbf{R}^3/G$ with G generated by the three elements

$$\begin{aligned} \alpha_1 : (x^1, x^2, x^3) &\longrightarrow (x^1 + 2\pi R_1, x^2, x^3) \\ \alpha_2 : (x^1, x^2, x^3) &\longrightarrow (x^1, x^2 + 2\pi R_2, x^3) \\ \alpha_3 : (x^1, x^2, x^3) &\longrightarrow (x^1, x^2, x^3 + 2\pi R_3) \end{aligned} \quad (44)$$

The first Bieberbach manifold with nontrivial holonomy is a quotient of \mathbf{R}^3 generated by α^i along with the element β ;

$$\beta : (x^1, x^2, x^3) \longrightarrow (-x^1, -x^2, x^3 + \pi R_3) . \quad (45)$$

so that $\beta^2 = \alpha_3$ and $\beta\alpha_i\beta^{-1} = \alpha_i^{-1}$ if $i \neq 3$. This leads to a space with holonomy \mathbf{Z}_2 . (We have not written the most general such manifold: the lattice in the 12 plane can be arbitrary and need not be rectangular.)

The \mathbf{Z}_4 example is similar to the \mathbf{Z}_2 case only now we take $R_1 = R_2$ and β acts as

$$\beta : (x^1, x^2, x^3) \longrightarrow (-x^2, x^1, x^3 + \frac{\pi}{2}R_3) . \quad (46)$$

and hence we have $\beta^4 = \alpha_3$, $\beta\alpha_1\beta^{-1} = \alpha_2$ and $\beta\alpha_2\beta^{-1} = \alpha_1^{-1}$.

Next we consider the \mathbf{Z}_3 case. Here one must fix $R_1 = R_2$ and start with a hexagonal lattice, so that the α_1 generator is modified to

$$\alpha_1 : (x^1, x^2, x^3) \longrightarrow (x^1 + \sqrt{3}\pi R_1, x^2 + \pi R_1, x^3) \quad (47)$$

while α_2, α_3 are unchanged. The generator β is now

$$\beta : (x^1, x^2, x^3) \longrightarrow \left(-\frac{1}{2}x^1 + \frac{\sqrt{3}}{2}x^2, -\frac{\sqrt{3}}{2}x^1 - \frac{1}{2}x^2, x^3 + \frac{2\pi}{3}R_3 \right) \quad (48)$$

which satisfies $\beta^3 = \alpha_3$, $\beta\alpha_1\beta^{-1} = \alpha_2^{-1}$ and $\beta\alpha_2\beta^{-1} = \alpha_1\alpha_2^{-1}$.

Next we consider the \mathbf{Z}_6 case. Again we must fix $R_1 = R_2$ and start with a hexagonal lattice, so that the α_1 generator is modified to

$$\alpha_1 : (x^1, x^2, x^3) \rightarrow (x^1 + \sqrt{3}\pi R_1, x^2 - \pi R_1, x^3) \quad (49)$$

The generator β is now

$$\beta : (x^1, x^2, x^3) \longrightarrow \left(\frac{1}{2}x^1 + \frac{\sqrt{3}}{2}x^2, -\frac{\sqrt{3}}{2}x^1 + \frac{1}{2}x^2, x^3 + \frac{\pi}{3}R_3 \right) . \quad (50)$$

and satisfies $\beta^6 = \alpha_3$, $\beta\alpha_1\beta^{-1} = \alpha_2^{-1}$ and $\beta\alpha_2\beta^{-1} = \alpha_1\alpha_2$.

The final case has holonomy $\mathbf{Z}_2 \times \mathbf{Z}_2$. In addition to the α_i (defined to generate a rectangular lattice) we introduce three additional generators

$$\begin{aligned} \beta_1 : (x^1, x^2, x^3) &\longrightarrow (x^1 + \pi R_1, -x^2 + \pi R_2, -x^3) \\ \beta_2 : (x^1, x^2, x^3) &\longrightarrow (-x^1 + \pi R_1, x^2 + \pi R_2, -x^3 + \pi R_3) \\ \beta_3 : (x^1, x^2, x^3) &\longrightarrow (-x^1, -x^2, x^3 + \pi R_3) \end{aligned} \quad (51)$$

which satisfy $\beta_i^2 = \alpha_i$, $\beta_i\alpha_j\beta_i^{-1} = \alpha_j^{-1}$ if $i \neq j$ and $\beta_1\beta_2\beta_3 = \alpha_1$.

It has been shown in [12] that there are no Killing spinors on a Bieberbach manifold with nontrivial holonomy. To see this one notes that a Killing spinor on a Bieberbach manifold will lift to a Killing spinor on the covering space \mathbf{R}^3 . However it must lift to a Killing spinor which is invariant under the group G .

In order to proceed we need to define a lift of the group G to a group $\tilde{G} \subset Spin(3)$ acting on the spinor bundle of \mathbf{R}^3 , *i.e.* for each generator g of G we must find an element $\tilde{g} \in \tilde{G}$ such that $\pi(\tilde{g}) = g$ and which preserves the relations of the group G . Here $\pi : Spin(3) \rightarrow SO(3)$ is the usual 2-1 map. As detailed in [12] for each group G there will generically be several choices for \tilde{G} and these correspond to different spin structures on the Bieberbach manifold.

Next we must ask that the Killing spinor is invariant. This leads to a condition

$$\tilde{g} \circ \epsilon(g \circ x) = \epsilon(x) \quad (52)$$

Since there is a unique spin bundle on the covering space we may choose a frame on \mathbf{R}^3 so that the Killing spinors are just constant spinors $\epsilon = \epsilon_0$. The condition (52) is simply that $\tilde{g} \circ \epsilon_0 = \epsilon_0$. We now note that all of the non-trivial Bieberbach manifolds contain a generator β which includes a rotation in some plane by an amount different from 2π . The lift of such a generator

is an element $\tilde{\beta} \in Spin(3)$ such that $\tilde{\beta} \neq 1$. Hence it is impossible to find a constant spinor ϵ_0 such that $\tilde{\beta} \circ \epsilon_0 = \epsilon_0$.

However if we turn on the flat gauge connection

$$A = \frac{1}{2R_3} dx^3 \quad (53)$$

then we can construct invariant spinors. To see this note that the Killing spinors on \mathbf{R}^3 , *i.e.* spinors which satisfy $\mathcal{D}_\mu^+ \epsilon = 0$, are now

$$\epsilon = e^{-\frac{x^3}{2R_3} \Gamma_5} \epsilon_0 \quad (54)$$

where ϵ_0 is a constant spinor. The invariance condition (52) is now

$$e^{\frac{x^3}{2R_3} \Gamma_5} \tilde{g} e^{-\frac{(g \circ x)^3}{2R_3} \Gamma_5} \epsilon_0 = \epsilon_0 \quad (55)$$

For the first four non-trivial Bieberbach manifolds the only non-trivial generator is β which acts as

$$x^3 \rightarrow x^3 + \theta R^3, \quad \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \rightarrow \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \quad (56)$$

where $\theta = \pi, 2\pi/3, \pi/4, \pi/3$ for the holonomies $\mathcal{H} = \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_4, \mathbf{Z}_6$ respectively. The corresponding lift to $Spin(3)$ of β is

$$\tilde{\beta} = \pm e^{\frac{\theta}{2} \Gamma_{12}} \quad (57)$$

where the choice of sign reflects a choice of spin structure on \mathcal{M}_3 . The invariance condition is now simply

$$\pm e^{\frac{\theta}{2} \Gamma_{12}} e^{-\frac{\theta}{2} \Gamma_5} \epsilon_0 = \epsilon_0 \quad (58)$$

This can be solved by choosing the spin structure corresponding to the plus sign and projecting onto constant spinors that satisfy

$$\Gamma_{03} \epsilon_0 = \epsilon_0 \quad (59)$$

For the \mathbf{Z}_2 case one can also find a Killing spinor with the spin structure corresponding to the the minus sign by taking $\Gamma_{03} \epsilon_0 = -\epsilon_0$.

A key property of the first four non-trivial Bieberbach manifolds that enables these Killing spinors to exist is that the lift of G to \tilde{G} is $\tilde{G} = U(1) \subset Spin(3)$. This allows the holonomy of the spinor induced by each generator to be canceled by the phase shift induced by the $U(1)$ gauge connection A . For

the final Bieberbach manifold, with holonomy $\mathbf{Z}_2 \times \mathbf{Z}_2$, \tilde{G} is not contained in a $U(1)$ subgroup of $Spin(3)$ [12] and hence the holonomies cannot be canceled. Thus there are no Killing spinors.

How did this work? In ordinary supergravity the gravitinos are sections of $T^*(\mathcal{M}) \otimes S(\mathcal{M})$, where $S(\mathcal{M})$ is a spinor bundle and $T^*(\mathcal{M})$ is the cotangent bundle. For the manifolds constructed above there are no Killing spinors, *i.e.* covariantly constant sections of $S(\mathcal{M})$. In the NMS the gravitinos are sections of $T^*(\mathcal{M}) \otimes S(\mathcal{M}) \otimes L(\mathcal{M})$ where $L(\mathcal{M})$ is an additional flat line bundle. The point is that there are covariantly constant sections of $S(\mathcal{M}) \otimes L(\mathcal{M})$.

Note that one might try to make the Bieberbach manifolds supersymmetric in old minimal supergravity by changing the boundary conditions to $\psi_\mu(x^3 + \theta R_3) = \Gamma_5 \psi_\mu(x^3)$ ⁴. Such a boundary condition is not compatible with the possible spin structures of spacetime but in principle this could be rectified by taking the Fermions to be sections of a line bundle associated to chiral rotations, as is the case in NMS, although without including a connection. However this approach is problematic as the supersymmetry variations of the auxiliary fields in old minimal supergravity are not chirally covariant.

This is reminiscent of the situation with spin^c structures. In these cases there are manifolds, for example \mathbf{CP}^2 , which don't admit any spinors at all, let alone covariantly constant ones. However they do admit sections of the spin bundle tensored with a complex line bundle; $S(\mathcal{M}) \otimes L(\mathcal{M})$ (*e.g.* see [13]). Indeed this situation can arise in string theory and M-theory [14]. Typically the complex line bundle is not flat and so cannot be a solution of NMS, at least without coupling to additional fields. However in NMS it is possible to include Spin^c manifolds in the off-shell formulation of the theory by taking the gauge field $F = dA$ to be non-vanishing and (cohomologically) non-trivial. In this sense \mathbf{CP}^2 is no more problematic than S^4 , *i.e.* the theory is defined for such manifolds but they do not satisfy the equations of motion.

5 Cylindrical Spacetimes and Singletons

Gauge degrees of freedom are typically thought of as unphysical. However this is not necessarily the case if spacetime has a boundary. Quite generally, putting a gauge theory on a spacetime with a spatial boundary can lead to physical gauge modes that live on the boundary. This happens, for example, in three-dimensional Chern-Simons gauge theory and in the theory of the fractional quantum Hall effect, where the boundary degrees of freedom are known as “edge states.” In the context of supergravity one can find some

⁴We thank J. Maldacena for discussion on this point.

discussion of these “singleton modes” in [15, 16, 17, 18]. Since NMS (and the variant (39)) has additional, but auxiliary, gauge degrees of freedom as compared with old minimal supergravity it is possible that one could in principle distinguish between them by considering spacetimes with a boundary. In some cases we may therefore hope to see residual gauge degrees of freedom propagating on the boundary. A particular class of spacetimes with a boundary are the so-called cylindrical spacetimes $\mathcal{M} = \mathcal{M}' \times \mathbf{R}$, where \mathbf{R} is the time dimension and $\partial\mathcal{M}' = \Sigma$. Thus the boundary of spacetime is $\Sigma \times \mathbf{R}$. Note that Σ could have several disconnected pieces.

We wish to show that, in NMS, there is a consistent choice of boundary conditions so that the theory on a boundary contains additional physical modes that propagate on the boundary due to the auxiliary fields. Hence one can, in principle, physically distinguish between different off-shell formulations of supergravity and, for example, determine the existence or non-existence of a given set of auxiliary fields.

5.1 An example

First it is helpful to review the discussion of appendix A in [16]. Consider a Bosonic action

$$S = \int_{\mathcal{M}} dA \wedge B \quad (60)$$

where $\mathcal{M} = \mathcal{M}' \times \mathbf{R}$ with $\partial\mathcal{M}' = \Sigma$.

To exhibit the singleton modes on the boundary we must carefully consider the boundary conditions to ensure a well posed variational problem. Expanding in terms of the temporal and spatial components of A and B this action is

$$S = \int dt \int_{\mathcal{M}'} d'A' \wedge B_0 + A_0 \wedge d'B' + \dot{A}' \wedge B' - \int dt \int_{\Sigma} A_0 \wedge B' \quad (61)$$

We can proceed in two ways. We could invoke the boundary condition $A_0 = 0$ on Σ . Alternatively we could simply add an additional boundary term to the theory

$$S_{\partial} = \int dt \int_{\Sigma} A_0 \wedge B' \quad (62)$$

to cancel the existing boundary term in (61). No boundary condition is now required on A_0 or B_0 . Presumably these two approaches are equivalent and in either case the gauge symmetry is broken on the boundary.

Continuing we can integrate out A_0 and B_0 since they are non-dynamical to find

$$d'A' = 0 \quad d'B' = 0 \quad (63)$$

which we solve by

$$A' = d'\Phi_A \quad B' = d'\Phi_B \quad (64)$$

where Φ_A and Φ_B are arbitrary. Substituting this back into the action leads to

$$\begin{aligned} S &= \int dt \int_{\mathcal{M}'} d'\dot{\Phi}_A \wedge d'\Phi_B \\ &= \int dt \int_{\Sigma} \dot{\Phi}_A \wedge d'\Phi_B \end{aligned} \quad (65)$$

Here we see the propagating singleton modes on the boundary.

One can think of these singleton modes as arising from pure gauge modes which violate the boundary condition $A_0 = 0$. To illustrate this point we note that in order to obtain a well-defined boundary value problem we can also choose the boundary condition

$$A = d\Phi_A, \quad dB = 0 \quad (66)$$

with Φ_A arbitrary, *i.e.* A is exact and B closed on $\partial\mathcal{M}$. Although it is important to note that such a boundary condition removes topologically non-trivial gauge configurations.

In this case no gauge symmetries are broken by the boundary. Let us proceed as above and integrate over the bulk A_0 and B_0 fields. By this we mean that we split $A_0 = a_0 + \tilde{A}_0$, where \tilde{A}_0 vanishes on $\partial\mathcal{M}$ and a_0 has support on $\partial\mathcal{M}$, and then integrate over \tilde{A}_0 . In this way we find

$$S = \int dt \int_{\Sigma} \dot{\Phi}_A \wedge d'\Phi_B - a_0 \wedge d'\Phi_B \quad (67)$$

Finally we observe that the boundary conditions imply that $a_0 = \dot{\Phi}_A$ and hence the boundary action vanishes. Thus, in this case, there are no boundary modes.

Finally we can consider what happens in the case where the theory is not quite topological but includes a standard kinetic term for B

$$S = \int_{\mathcal{M}} dA \wedge B + \frac{1}{2} dB \wedge \star dB \quad (68)$$

It is not longer so simple to integrate out B_0 . However we can see that if we choose the boundary conditions which break the gauge symmetries then there will be massless gauge modes that propagate along the boundary. These can also be thought of as Goldstone modes for the global symmetry resulting from gauge transformations which do not vanish on the boundary. For further details on singleton modes in such theories see [17, 18, 19].

5.2 Singletons in NMS

NMS and its variant (39) contain the same $dA \wedge B$ coupling that we have just discussed. (In NMS there is also a kinetic term for B which is absent in the latter case.) Therefore we expect that if we choose gauge symmetry violating boundary conditions then singleton modes will propagate along the boundary.

We saw that by adding a suitable boundary term we could ensure that both these actions were supersymmetric on a manifold with boundary provided that we imposed the correct boundary conditions and terms. In particular we required that $A = d\Phi_A$ and $V = 0$ on $\partial\mathcal{M}$. These boundary conditions restrict the topology of the connection A but preserve the gauge symmetry in the presence of the boundary. Thus there will not be any singleton modes in this case.

In the more interesting case that we do not want to, or cannot, restrict the gauge field A to be exact then supersymmetry will be broken by the boundary. Furthermore given the previous discussion we expect to see singleton modes. In the case of NMS there is a kinetic term for B , just as in the action (68). However from the discussion of (68) it is clear that there will be singleton modes from the gauge symmetry if we impose the boundary condition $A_0 = 0$ on $\partial\mathcal{M}$.

In the case of the variant theory (39) there is no kinetic term for B and we can be more explicit. In particular we choose the boundary condition $A_0 = 0$. Proceeding as before we can integrate out A_0 which leads to the constraint

$$d'B' - i\bar{\psi}_i \Gamma_j \psi_k dx^i \wedge dx^j \wedge dx^k = 0 \quad (69)$$

Next we integrate out B_0 to find

$$d'A' = 0 \quad (70)$$

Thus we can set $A' = d'\Phi_A$. Substituting all this back into the action we find

$$S = \int dt \int_{\mathcal{M}'} \sqrt{-g} R - 4i\bar{\psi}_i \Gamma^{i0k} D_0 \psi_k - 4i\bar{\psi}_\mu \Gamma^{\mu\nu} \mathcal{D}_i^+ \psi_\nu - 4d'\dot{\Phi}_A \wedge B' \quad (71)$$

Note that D_0 appears instead of \mathcal{D}_0^+ . Next we integrate the last term by parts and use the constraint (69)

$$S = \int dt \int_{\mathcal{M}'} \sqrt{-g} R - 4i\bar{\psi}_\mu \Gamma^{\mu\nu\lambda} e^{-\Phi_A \Gamma_5} D_\nu (e^{\Phi_A \Gamma_5} \psi_\lambda) - 4 \int dt \int_\Sigma \dot{\Phi}_A \wedge B' \quad (72)$$

Lastly we perform the field redefinition $\psi_\mu = e^{-\Phi_A \Gamma_5} \Psi_\mu$ and arrive at the familiar on-shell supergravity but with an additional boundary term

$$S = \int dt \int_{\mathcal{M}'} \sqrt{-g} R - 4i\bar{\Psi}_\mu \Gamma^{\mu\nu\lambda} D_\nu \Psi_\lambda - 4 \int dt \int_\Sigma \dot{\Phi}_A \wedge B' \quad (73)$$

We must still solve for the constraint (69), be precise about the Fermionic boundary conditions and include any appropriate boundary terms, such as the Gibbons-Hawking term. However regardless of how we do this it is clear that we will always have $B' = d\Phi_B + \dots$ where Φ_B is a Bosonic boundary mode and the ellipsis denotes Fermionic terms. For example if we assume that $A_0 = B_0 = 0$, $\Psi_\mu = D_\mu\eta$ is pure gauge and $R_{\mu\nu\lambda\rho} = 0$ on the boundary then we have a well posed boundary value problem and we find that

$$B' = d'\Phi_B + i\bar{\eta}\Gamma_i D_j \eta dx^i \wedge dx^j \quad (74)$$

on $\partial\mathcal{M}$ so that the Fermionic gauge modes also propagate along the boundary. Note that as a consequence of their topological origin the singleton modes do not come with a factor of $\sqrt{-g}$ and hence do not contribute to Einstein's equation.

5.3 Supersymmetry transformation of Φ_B

It is helpful to consider the on-shell supersymmetries of NMS. These are

$$\begin{aligned} \delta_\epsilon e_\mu{}^\nu &= -2i\bar{\epsilon}\Gamma^\nu\psi_\mu \\ \delta_\epsilon\psi_\mu &= \mathcal{D}_\mu^+\epsilon \\ \delta_\epsilon B_{\mu\nu} &= 4i\bar{\epsilon}\Gamma_{[\mu}\psi_{\nu]} \\ \delta_\epsilon A_\mu &= 0 \end{aligned} \quad (75)$$

and one can check that they preserve the on-shell conditions $dA = V = 0$, as they should. Note that from the condition $V = 0$ we must have

$$dB - i\bar{\psi}_\nu\Gamma_\lambda\psi_\rho dx^\nu \wedge dx^\lambda \wedge dx^\rho = 0 \quad (76)$$

If $\psi_\mu = \mathcal{D}_\mu^+\eta$ is pure gauge then

$$dB = id\left(\bar{\eta}\Gamma_\lambda\mathcal{D}_\rho^+\eta dx^\lambda \wedge dx^\rho\right) - i\bar{\eta}\Gamma_\lambda\mathcal{D}_\nu^+\mathcal{D}_\rho^+\eta dx^\nu \wedge dx^\lambda \wedge dx^\rho \quad (77)$$

The second term will vanish on-shell so that

$$B = d\Phi_B + i\bar{\eta}\Gamma_\lambda\mathcal{D}_\rho^+\eta dx^\lambda \wedge dx^\rho \quad (78)$$

for an arbitrary one-form Φ_B . Under a supersymmetry generated by ϵ we clearly have that

$$\delta_\epsilon\eta = \epsilon \quad (79)$$

Using the expression above for $\delta_\epsilon B_{\mu\nu}$ we see that

$$\begin{aligned} 2i\bar{\epsilon}\Gamma_\mu\mathcal{D}_\nu^+\eta\,dx^\mu\wedge dx^\nu &= d\delta_\epsilon\Phi_B + (i\bar{\epsilon}\Gamma_\lambda\mathcal{D}_\rho^+\eta + i\bar{\eta}\Gamma_\lambda\mathcal{D}_\rho^+\epsilon)dx^\lambda\wedge dx^\rho \\ &= d(\delta_\epsilon\Phi_B + i\bar{\epsilon}\Gamma_\lambda\eta dx^\lambda) + 2i\bar{\epsilon}\Gamma_\mu\mathcal{D}_\nu^+\eta\,dx^\mu\wedge dx^\nu \end{aligned} \quad (80)$$

Thus

$$\delta_\epsilon\Phi_B = -i\bar{\epsilon}\Gamma_\mu\eta dx^\mu \quad (81)$$

Hence we see that the gauge zero modes η and Φ_B are related by supersymmetry.

If we are on a manifold with boundary and use the boundary conditions (26) then the preserved supersymmetry is ϵ_+ and we must set $\eta_- = 0$ on $\partial\mathcal{M}$, where the signs denote the eigenvalue of $\xi\Gamma^n e^{2\Phi_A\Gamma^5}$. In this case we see that

$$\delta_\epsilon\Phi_B = -i\bar{\epsilon}_+\Gamma_{\mu'}\eta_+ dx^{\mu'} \quad (82)$$

Thus only the component of Φ_B that is tangential to the boundary is related to η_+ by supersymmetry.

6 Discussion

In this paper we have discussed various aspects of the auxiliary fields that arise in new minimal supergravity (NMS). In particular we showed that there are compact three-manifolds with well-defined Killing spinors which are not well-defined in old minimal supergravity or simple off-shell supergravity. We also showed that, subject to suitable boundary conditions, the auxiliary fields actually give rise to physical on-shell degrees of freedom that reside on the boundary of spacetime. Thus one can in principle distinguish between different off-shell forms of supergravity using on-shell physics. We also demonstrated how half of the supersymmetry could be preserved in NMS on a manifold with boundary, provided the gauge field is trivial on the boundary. This suggests that there might be interesting applications to brane world scenarios where a topologically non-trivial auxiliary gauge field would lead to supersymmetry breaking. Finally we would like to address some related issues.

It would be worthwhile extending the discussion of the present paper to other off-shell formulations of supergravity. Apart from old and new minimal supergravity there is also the so-called β FFC formulation [20]. It was observed in [21] that the β FFC formulation can be understood as the coupling of NMS supergravity to a compensating chiral multiplet whose Bosonic content is a complex scalar (along with an auxiliary field). The logarithm of the

absolute value of the scalar field is identified with the dilaton ϕ whereas its phase is eaten by the two-form B to produce a dynamical two-form which is dualized to the axion a .

Let us describe the β FFC formulation in more detail. The complex scalar of the compensating chiral multiplet is given a non-vanishing chiral weight. In particular, under a chiral transformation, its phase φ is shifted; $\varphi \rightarrow \varphi - \chi$ while its absolute value is invariant. The chiral covariant derivative of φ is therefore

$$\mathcal{D}_\mu^+ \varphi = \partial_\mu \varphi + A_\mu \quad (83)$$

Hence the kinetic term for φ introduces a quadratic term for the chiral gauge field A in the Lagrangian. The resulting A equation of motion now algebraically determines A in terms of B and φ to be [21]

$$A = -d\varphi - 4 \star dB \quad (84)$$

(Recall that without coupling to the compensating chiral multiplet the A equation of motion ensured that B was non-dynamical: $dB = 0$.) Thus in the β FFC formulation there is still a chiral gauge field that couples minimally to the Fermions, only now it is determined by B and φ . Note that the equation of motion for B is $d(e^{2\phi} d \star B) = 0$ and hence it is possible to have $dA \neq 0$ on-shell.

However we cannot make the Bieberbach manifolds supersymmetric as we did for NMS since if $A = dx^3/2R$ then we must have $\varphi = x^3/2R$ or $dB \neq 0$. The former case is forbidden as there are couplings of φ to the Fermions in the Lagrangian which require that φ be single valued. In the latter case one sees that a non-zero dB will lead to a non-vanishing energy-momentum tensor so that the Bieberbach manifolds will no longer satisfy the Einstein equations (although this raises the possibility of interesting new supersymmetric “flux compactifications”).

It is important to observe that the chiral symmetry that NMS supergravity gauges is anomalous. This has been shown [22] to lead to supersymmetry anomalies in the quantum theory. Happily all is not lost as a Green-Schwarz anomaly cancelation for NMS supergravity has been found in [23, 24] and one can show that the Bieberbach manifolds remain supersymmetric.

There has been some debate in the literature as to whether or not old minimal, NMS or the β FFC formulation results from four-dimensional string theory [25, 26, 27, 28, 21] (see also [29, 30] for related discussions on the appearance of new minimal and the β FFC formulations). However the main message of this paper has been to show that there can be hidden on-shell physics in the auxiliary fields and these remain largely unknown in higher

dimensions. Therefore to make further contact with string theory it is important to develop NMS and other off-shell formulations further. In particular it is not clear how to couple NMS to chiral multiplets with a potential, as is needed in string theory. The problem is that the superpotential must transform under chiral symmetries. One way to achieve this might be to postulate a chiral multiplet with a complex scalar ϕ_0 that shifts under the chiral symmetry

$$\delta_\chi \phi_0 = -\chi \quad (85)$$

in addition to the other scalars ϕ_I which are chirally invariant. Therefore the covariant derivative acts on ϕ_0 as

$$\mathcal{D}_\mu^+ \phi_0 = D_\mu \phi_0 + A_\mu \quad (86)$$

and is the ordinary derivative on ϕ_I . If this could be incorporated into NMS then one could attempt to include couplings to a superpotential of the form

$$W = e^{2i\phi_0} \tilde{W}(\phi_I) \quad (87)$$

where \tilde{W} depends holomorphically on ϕ_I . This suggestion is reminiscent of the β FFC formulation, coupled to a superpotential. Therefore one expects similar effects whereby B eats the real part of ϕ_0 and becomes the dynamical axion and the auxiliary gauge field A_μ is algebraically determined in terms of the other fields.

Finally, let us return to our motivation of formulating M -theory on Spin^c manifolds. Of course, we do not want to introduce a new propagating degree of freedom through the Spin^c connection. In [14] this degree of freedom is part of the B -field, but it is not evident how to implement such a relation in general. Another problem one must face is reconciling the Spin^c structure with the standard reality properties of the gravitino. Finally, anomaly cancellation arguments would need to be modified. For example, the quantization of G -flux of [31] now becomes $[G]_{DR} = (\frac{1}{4}\bar{p}_1 + \frac{1}{2}(\bar{c}_1)^2) \bmod \bar{H}^4(Y, Z)$, where the overline denotes reduction modulo torsion, and c_1 is the Chern class of the Spin^c structure. (See [32] for related discussion.) Thus, finding such a generalization of M -theory - if it exists - seems quite challenging. The results of this paper make it clear that in such a search, one must first decide on some choice of off-shell formulation of the theory.

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